Math 10B with Professor Stankova
Worksheet, Discussion #17; **Thursday, 3/21/2019**
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Improper Integrals and Histograms

Concepts

1. An **improper integral** is an integral where one or two of the bounds of integration are $\pm\infty$. If both bounds are $\pm\infty$, we first need to split up the integral at some finite point (usually we pick 0 for convenience). Then to compute a one-sided improper integrals, we first write it as a limit as $n \to \infty$ then compute the limit. If any improper integral is ∞, we say the integral does not exist (even in the case of two sided integrals we get $\infty - \infty$).

   For a histogram, the area of the rectangle represents the probability of lying inside that region. So, if there is a 10% chance of X being between 10 and 15, the height of the rectangle would be $0.1/(15 - 10) = 0.02$.

Example

2. Calculate $\displaystyle\int_{-\infty}^{\infty} e^{-|x|}dx$.

3. Suppose exam scores are given by the following data. Construct a histogram of the data with intervals $[40, 50), [50, 60), \dots, [90 - 100)$.

Score	40-49	50-59	60-69	70-79	80-89	90-99
Frequency	3	2	8	4	2	1

Problems

4. True False When calculating $\displaystyle\int_{-\infty}^{\infty} f(x)dx$, the final result depends on the a we choose to split it up as $\displaystyle\int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx$.

5. True False Since the function $f(x) = x$ is odd, we know that $\displaystyle\int_{-n}^{n} f(x)dx = 0$ for all integers n.

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7. Compute $\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx$.

8. Draw a new histogram using the previous data except with intervals $[40, 60)$, $[60, 80)$, $[80, 100)$.

Continuous Probability

Concepts

9. When we deal with random variables that can take a continuum of values, we have to use PDFs as opposed to PMFs. In this case, the value of the PMF does not give you a probability of picking that value, but instead gives you a relative likelihood. So if $f(x) = 2f(y)$, where f is the PDF, you expect to get a value near x around twice as likely as you are to get a value near y. Another difference from PMFs is now to calculate probabilities, we must take the integral along the interval we are asking about. So $P(a \leq X \leq b) = \int_a^b f(x)dx$. The most important property of a PDF is $\boxed{\int_{-\infty}^{\infty} f(x)dx = 1}$.

   A CDF is a function $F(x)$ where $F(x) = P(X \leq x)$, it tells us that probability of getting a value less than or equal to x. It is just defined as $F(x) = \int_{-\infty}^{x} f(x)dx$. It satisfies three important properties:

   - $F(x)$ is nondecreasing. So if $x \leq y$, then $F(x) \leq F(y)$.
   - $\lim\limits_{x\to-\infty} F(x) = 0$.
   - $\lim\limits_{x\to\infty} F(x) = 1$.

Example

10. Suppose that the probability density function P that an atom emits a gamma wave with the PDF $P(t) = Ce^{-10t}$ for $t \geq 0$ and $P(t) = 0$ for $t < 0$. Find P and calculate the CDF associated with P.

11. For the above PDF, find the probability that a gamma wave is emitted from -1 seconds to 1 second.

Problems

12. True False Since the CDF is an antiderivative of the PDF, there are multiple CDFs for a given PDF (and they differ by a $+C$).

13. True False The area underneath a CDF must be equal to 1.

14. True False A PDF must be continuous.

15. True False Let $P(x) = Cx^3$ for $-1 \leq x \leq 2$ and 0 otherwise. Since $\int_{-1}^{3} P(x)dx = C(16 - 1/4)$, setting $C = (16 - 1/4)^{-1}$ makes P into a PDF.

16. Let $P(x) = Cx^2(10 - x)$ on $0 \leq x \leq 10$ and 0 otherwise. Find C such that P is a PDF and its corresponding CDF. Find the probability that we choose a number between 0 and 1.

17. Let $P(x) = C(x - 1)(x + 1)$ on $-1 \leq x \leq 1$ and 0 otherwise. Find C such that P is a PDF and its corresponding CDF. Find the probability that we choose a number between 0 and 1.

18. Let $P(x)$ satisfy $\frac{dP}{dx} = 2x$ for $0 \leq x \leq 1$ and $P(x) = 0$ otherwise. Find P such that it is a PDF and its corresponding CDF. Find the probability that we choose a number between 0 and 1.

19. Let $F(x) = \frac{x-1}{x+1}$ for $x \geq 1$ and 0 for $x \leq 1$. Show that F is a CDF. Find the PDF associated with it and the probability that we choose a number between 1 and 2.

20. Find numbers A, B such that $A \arctan(x) + B$ is a CDF and find the PDF associated with it. Find the probability that we choose a number between 0 and 1.

21. Let $F(x) = \ln x$ for $1 \leq x \leq a$ and $F(x) = 0$ for $x \leq 1$ and $F(x) = 1$ for $x \geq a$. Find a such that F is a continuous CDF and find the PDF associated with it. Find the probability that we choose a number between 1 and 2.